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Summation Formulas Associated with the Lauricella Function $F_A^{(r)}$

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Abstract—The authors make use of some rather elementary techniques in order to derive several summation formulas associated with Lauricella's hypergeometric function $F_A^{(r)}$ in r variables (and also with its familiar generalizations). A number of (known or new) consequences of these summation formulas are also considered.

Keywords—Lauricella functions, Multiple hypergeometric functions, Appell functions, Laguerre polynomials, Confluent hypergeometric function, Gauss hypergeometric function, Cauchy product, Laplace transforms, Generating functions.

1. INTRODUCTION AND DEFINITIONS

Multiple hypergeometric functions (that is, hypergeometric functions in several variables) occur naturally in a wide variety of problems (see, for details, [1, p. 47 *et seq.*, Section 1.7]). In particular, the Lauricella function $F_A^{(r)}$ in r variables, defined by (cf. [2]; see also, [1, p. 33, equation 1.4(1)])

$$\begin{gathered} F_A^{(r)}\left[a, b_1, \dots, b_r;\ c_1, \dots, c_r;\ x_1, \dots, x_r\right] \\ := \sum_{m_1,\dots,m_r=0}^{\infty} \frac{(a)_{m_1+\cdots+m_r}(b_1)_{m_1}\cdots(b_r)_{m_r}}{(c_1)_{m_1}\cdots(c_r)_{m_r}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!} \qquad (1) \\ \left(|x_1| + \cdots + |x_r| < 1;\ (\lambda)_m := \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)}\right) \end{gathered}$$

together with its special case when $r = 2$ (namely, the Appell function F_2) arise frequently in various physical and quantum chemical applications (cf., e.g., [1, pp. 49–50 and 293]).

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Motivated essentially by the aforementioned occurrences of multiple hypergeometric functions, we aim here at applying some rather elementary techniques with a view to deriving a number of summation formulas associated with the Lauricella function $F_A^{(r)}$ and its familiar generalizations. We also consider several (known or new) consequences of these summation formulas.

2. A FINITE SUM INVOLVING $F_A^{(r)}$

The classical Laguerre polynomials $L_n^{(\alpha)}(x)$, given explicitly by [3, p. 220, equation 112(1)]

$$L_n^{(\alpha)}(x) = \binom{\alpha+n}{n} {}_1F_1(-n;\alpha+1;x) \tag{2}$$

in terms of the (Kummer's) confluent hypergeometric function ${}_1F_1$, are known to possess the generating function [3, p. 202, equation 113(4)]:

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n = (1-t)^{-\alpha-1} \exp\left(-\frac{xt}{1-t}\right), \qquad (|t|<1). \tag{3}$$

Replacing α in (3) by $\alpha+\beta+1$, we readily obtain

$$\sum_{n=0}^{\infty} L_n^{(\alpha+\beta+1)}(x)\, t^n = \left(\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n\right)\left(\sum_{n=0}^{\infty} L_n^{(\beta)}(x)\, t^n\right), \tag{4}$$

which, upon interpreting the right-hand side as a Cauchy product, immediately yields the following well-known identity for the Laguerre polynomials (cf., e.g., [4, p. 321, Entry (48.21.4)], where many *earlier* references are also provided):

$$\sum_{k=0}^{n} L_k^{(\alpha)}(x)\, L_{n-k}^{(\beta)}(y) = L_n^{(\alpha+\beta+1)}(x+y). \tag{5}$$

Next, we recall an interesting integral representation for the Lauricella function $F_A^{(r)}$ in the form [1, p. 285, equation 9.4(35)]:

$$\begin{gathered} F_A^{(r)}\left[a, b_1, \ldots, b_r;\ c_1, \ldots, c_r;\ x_1, \ldots, x_r\right] \\ = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-t} t_1^{a-1} {}_1F_1\begin{bmatrix} b_1; & x_1 t \\ c_1; & \end{bmatrix} \cdots {}_1F_1\begin{bmatrix} b_r; & x_r t \\ c_r; & \end{bmatrix} dt, \\ (\Re(x_1+\cdots+x_r)<1;\ \Re(a)>0), \end{gathered} \tag{6}$$

which clearly exhibits the fact that the Laplace transform of a product of r ${}_1F_1$ functions can be expressed in terms of the Lauricella function $F_A^{(r)}$.

In view of the ${}_1F_1$ representation (2) for the Laguerre polynomials, it is easily seen from (5) (with, of course, $\alpha=\beta=0$) and (6) that

$$\begin{gathered} \sum_{k=0}^{n} F_A^{(r)}\left[a, -k, -n+k, b_3, \ldots, b_r; 1, 1, c_3, \ldots, c_r; x_1, \ldots, x_r\right] \\ = (n+1) F_A^{(r-1)}\left[a, -n, b_3, \ldots, b_r; 2, c_3, \ldots, c_r; x_1+x_2, x_3, \ldots, x_r\right], \\ (|x_1|+\cdots+|x_r|<1), \end{gathered} \tag{7}$$

which expresses a finite sum of a special Lauricella function $F_A^{(r)}$ as a lower-order Lauricella function $F_A^{(r-1)}$.

For the Appell function F_2, a special case of (7) when $r = 2$ yields the finite summation formula:

$$\sum_{k=0}^{n} F_2[a, -k, -n+k; 1, 1; x, y] = (n+1)_2 F_1(a, -n; 2; x+y), \tag{8}$$

which is presumably new. We remark in passing that (8) cannot be derived from the special case $\alpha = 0$ of the following known summation formula due to Srivastava [5, p. 1088, equation (1.5)]:

$$\begin{aligned}&\sum_{k=0}^{n} \binom{\alpha+k}{k} F_2[a, -k, -k; \alpha+1, \alpha+1; x, y] \\ &= \frac{(\alpha+1)_{n+1}}{n!\,(a-1)} (x-y)^{-1} F_2[a-1,\ -n,\ -n-1;\ \alpha+1,\ \alpha+1;\ x, y] + x \leftrightarrow y,\end{aligned} \tag{9}$$

where $x \leftrightarrow y$ indicates the presence of a second term which originates from the first by interchanging x and y (see also [6–8] for *further* generalizations of (9) involving two and three variables, and Section 4 below for its multivariable generalizations).

For $x = y = 1/2$, the Gauss hypergeometric function occurring on the right-hand side of (8) can be evaluated by appealing to the Chu-Vandermonde summation theorem [3, p. 69, Exercise 4], and we thus have the summation formula:

$$\sum_{k=0}^{n} F_2\left[a,\ -k,\ -n+k;\ 1,\ 1;\ \frac{1}{2}, \frac{1}{2}\right] = \frac{(2-a)_n}{n!}, \tag{10}$$

which, for $a = 1$, assumes the elegant form:

$$\sum_{k=0}^{n} F_2\left[1,\ -k,\ -n+k;\ 1,\ 1;\ \frac{1}{2}, \frac{1}{2}\right] = 1. \tag{11}$$

By iterating the method of derivation of the summation formula (7), based upon the identity (5) and the integral representation (6), we can obtain the following generalization of (7):

$$\begin{aligned}&\sum_{k_1=0}^{n_1} \cdots \sum_{k_s=0}^{n_s} F_A^{(2s)}[a,\ -k_1,\ -n_1+k_1, \ldots, -k_s,\ -n_s+k_s; 1, \ldots, 1; x_1, \ldots, x_{2s}] \\ &= (n_1+1)\ldots(n_s+1) F_A^{(s)}[a,\ -n_1, \ldots, -n_s;\ 2, \ldots, 2; x_1+x_2,\ x_3+x_4, \ldots, x_{2s-1}+x_{2s}],\end{aligned} \tag{12}$$

which, for $s = 1$, corresponds to (8).

3. THE LAURICELLA FUNCTION $F_A^{(2s)}$ AS AN s-FOLD SUM

By interpreting the first two ${}_1F_1$ functions occurring on the right-hand side of (6) as a Cauchy product, it is easily seen from (6) that

$$\begin{aligned}&F_A^{(r)}[a, b_1, \ldots, b_r; c_1, \ldots, c_r; x_1, \ldots, x_r] \\ &= \sum_{m=0}^{\infty} \frac{(a)_m (b_1)_m}{(c_1)_m} \frac{x_1^m}{m!}\, {}_3F_2\left[\begin{matrix}-m, & 1-c_1-m, & b_2; \\ & 1-b_1-m, & c_2;\end{matrix}\ -\frac{x_2}{x_1}\right] \\ &\cdot F_A^{(r-2)}[a+m, b_3, \ldots, b_r; c_3, \ldots, c_r; x_3, \ldots, x_r] \\ &(|x_1| + \cdots + |x_r| < 1),\end{aligned} \tag{13}$$

which expresses the Lauricella function $F_A^{(r)}$ as a series of the lower-order Lauricella functions $F_A^{(r-2)}$.

More generally, by iterating the above process, (6) would finally yield

$$\begin{aligned} &F_A^{(2s)}\left[a,\, b_1,\dots,b_{2s};\; c_1,\dots,c_{2s};\; x_1,\dots,x_{2s}\right] \\ &= \sum_{m_1,\dots,m_s=0}^{\infty} (a)_{m_1+\dots+m_s} \prod_{j=1}^{s} \left\{ \frac{(b_{2j-1})_{m_j}}{(c_{2j-1})_{m_j}} \frac{x_{2j-1}^{m_j}}{m_j!} \right. \\ &\left. \cdot {}_3F_2\left[\begin{matrix} -m_j, & 1-c_{2j-1}-m_j, & b_{2j}; \\ & 1-b_{2j-1}-m_j, & c_{2j}; \end{matrix} -\frac{x_{2j}}{x_{2j-1}}\right] \right\}, \\ &(|x_1|+\dots+|x_{2s}|<1), \end{aligned} \tag{14}$$

which indeed provides the desired s-fold sum for the Lauricella function $F_A^{(2s)}$.

In its special case when $r = 2$, the summation formula (13) can easily be rewritten in terms of the Appell function F_2. We thus obtain the known result [9, p. 181, problem 38(ii)]:

$$\begin{aligned} F_2\left[\alpha,\beta,\beta';\gamma,\gamma';x,y\right] &= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m} \frac{x^m}{m!} \left[\begin{matrix} -m, & 1-\gamma-m, & \beta'; \\ & 1-\beta-m, & \gamma'; \end{matrix} -\frac{y}{x}\right], \\ &(|x|+|y|<1), \end{aligned} \tag{15}$$

which is contained in a much more general formula [9, p. 156, equation 2.6(58)] in the theory of generating functions.

4. MULTIVARIABLE EXTENSIONS OF THE SUMMATION FORMULA (9)

In terms of the Lauricella function $F_A^{(r)}$, Srivastava's summation formula (9) can easily be generalized in the form:

$$\begin{aligned} &\sum_{k=0}^{n} \binom{\alpha+k}{k} F_A^{(r)}\left[a,-k,-k,b_3,\dots,b_r;\alpha+1,\alpha+1,c_3,\dots,c_r;x_1,\dots,x_r\right] \\ &= \frac{(\alpha+1)_{n+1}}{n!(a-1)} (x_1-x_2)^{-1} F_A^{(r)}\left[a-1,-n,-n-1,b_3,\dots,b_r;\right. \\ &\qquad \left. \alpha+1,\alpha+1,\, c_3,\dots,c_r;x_1,\dots,x_r\right] + x_1 \leftrightarrow x_2, \end{aligned} \tag{16}$$

where, just as in (9), $x_1 \leftrightarrow x_2$ indicates the presence of a second term which originates from the first by interchanging x_1 and x_2.

For $x_j = 0$ $(j = 3,\dots,r)$, equation (16) reduces immediately to Srivastava's summation formula (9). Moreover, (16) with $r = 3$ is substantially the same as another generalization of (9) by Srivastava [8, p. 318, equation (3.2)].

By suitably iterating the above process of deriving (16) from Srivastava's summation formula (9) a couple of times, we obtain

$$\begin{aligned} &\sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \sum_{k_3=0}^{n_3} \binom{\alpha_1+k_1}{k_1} \binom{\alpha_2+k_2}{k_2} \binom{\alpha_3+k_3}{k_3} \\ &\times F_A^{(r)}\left[a,-k_1,-k_1,-k_2,-k_2,-k_3,-k_3,b_7,\dots,b_r;\right. \\ &\left. \alpha_1+1,\alpha_1+1,\alpha_2+1,\alpha_2+1,\alpha_3+1,\alpha_3+1,c_7,\dots,c_r;\; x_1,\dots,x_r\right] \\ &= \frac{(\alpha_1+1)_{n_1+1}(\alpha_2+1)_{n_2+1}(\alpha_3+1)_{n_3+1}}{n_1!\; n_2!\; n_3!(a-1)(a-2)(a-3)} (x_1-x_2)^{-1} \\ &\times \left[(x_3-x_4)^{-1} \left\{(x_5-x_6)^{-1} F_A^{(r)}\left[a-3,-n_1,-n_1-1,-n_2,-n_2-1,-n_3,\right.\right.\right. \\ &-n_3-1,\, b_7,\dots,b_r;\alpha_1+1,\alpha_1+1,\alpha_2+1,\alpha_2+1,\alpha_3+1,\alpha_3+1,\, c_7,\dots,c_r; \\ &\left.\left.\left. x_1,\dots,x_r\right] + x_5 \leftrightarrow x_6 \right\} + x_3 \leftrightarrow x_4 \right] + x_1 \leftrightarrow x_2, \end{aligned} \tag{17}$$

where the right-hand side obviously has a total of 2^3 terms.

A considerably more involved expression can, of course, be derived similarly for the multiple sum:

$$\sum_{k_1=0}^{n_1} \cdots \sum_{k_s=0}^{n_s} \binom{\alpha_1+k_1}{k_1} \cdots \binom{\alpha_s+k_s}{k_s}$$

$$\cdot F_A^{(2s)}\left[a, -k_1, -k_1, \ldots, -k_s, -k_s; \alpha_1+1, \alpha_1+1, \ldots, \alpha_s+1, \alpha_s+1; x_1, \ldots, x_{2s}\right]$$

for $s \in \mathbb{N} \setminus \{1,2,3\}$. The details are being omitted here.

Finally, for a certain member of the class of the (Srivastava-Daoust) generalized Lauricella functions in r variables (cf., e.g., [1, p. 38, equation 1.4(24)]), we can derive the following further generalization of the summation formula (16) by resorting to the method of multidimensional mathematical induction using (16) in conjunction with the Laplace and inverse Laplace transforms (see, for details, [7,8]):

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{\alpha+k}{k} F^{p:1;1;u_3;\ldots;u_r}_{q:1;1;v_3;\ldots;v_r} \\
&\left[\begin{matrix} (\lambda_p): & -k; & -k; & \left(\rho^{(3)}_{u_3}\right); \ldots; \left(\rho^{(r)}_{u_r}\right); \\ (\mu_q): & \alpha+1; & \alpha+1; & \left(\sigma^{(3)}_{v_3}\right); \ldots; \left(\sigma^{(r)}_{v_r}\right); \end{matrix} \; x_1, \ldots, x_r\right] \\
&= \frac{(\alpha+1)_{n+1}}{n!\,(x_1-x_2)} \frac{\prod_{j=1}^{q} (\mu_j-1)}{\prod_{j=1}^{p} (\lambda_j-1)} F^{p:1;1;u_3;\ldots;u_r}_{q:1;1;v_3;\ldots;v_r} \\
&\left[\begin{matrix} (\lambda_p)-1: & -n; & -n-1; & \left(\rho^{(3)}_{u_3}\right); \ldots; \left(\rho^{(r)}_{u_r}\right); \\ (\mu_q)-1: & \alpha+1; & \alpha+1; & \left(\sigma^{(3)}_{v_3}\right); \ldots; \left(\sigma^{(r)}_{v_r}\right); \end{matrix} \; x_1, \ldots, x_r\right] + x_1 \leftrightarrow x_2,
\end{aligned} \tag{18}$$

where, for convenience, (λ_p) abbreviates the array of p parameters

$$\lambda_1, \ldots, \lambda_p,$$

and $\left(\rho^{(j)}_{u_j}\right)$ abbreviates the array of u_j parameters

$$\rho^{(j)}_1, \ldots, \rho^{(j)}_{u_j}; \qquad (j = 3, \ldots, r),$$

with similar interpretations for (μ_q) and $\left(\sigma^{(j)}_{v_j}\right)$ $(j = 3, \ldots, r)$.

Remark 1. The summation formula (16) is indeed recoverable from the special case of (18) when

$$p = q + 1 = u_j = v_j = 1, \qquad (j = 3, \ldots, r).$$

Remark 2. For $r = 2$, the summation formula (18) corresponds to a result due to Srivastava [7, p. 3, equation (2.1)].

Remark 3. A *three-variable* ($r = 3$) case of the summation formula (18) is related rather closely to another result of Srivastava [8, p. 315, equation (2.1)], which (in turn) yields the *main* result of Pathan [6, p. 785, equation (2.3)] as a very specialized case.

REFERENCES

1. H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press, Chichester, (1985).
2. G. Lauricella, Sulle funzioni ipergeometriche a più variabili, *Rend. Circ. Mat. Palermo* **7**, 111–158, (1893).
3. E.D. Rainville, *Special Functions*, Macmillan, New York, 1960; Reprinted by Chelsea Publishing, Bronx, NY, 1971.
4. E.R. Hansen, *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, NJ, (1975).
5. H.M. Srivastava, On a summation formula for the Appell function F_2, *Proc. Cambridge Philos. Soc.* **63**, 1087–1089, (1967).
6. M.A. Pathan, A note on the generalization of the summation formulas for Appell's and Kampé de Fériet's hypergeometric functions, *J. Math. Anal. Appl.* **72**, 783–787, (1979).
7. H.M. Srivastava, Some finite summation formulas involving generalized hypergeometric functions, *Indian J. Math.* (Prof. C.T. Rajagopal Memorial Volume) **22**, 1–7, (1980).
8. H.M. Srivastava, A class of finite summation formulas involving generalized hypergeometric functions, *Boll. Un. Mat. Ital. A* **5** (6), 311–320, (1986).
9. H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press, Chichester, (1984).